Economic and Econometrics Models

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Economic & Econometric Models

A model from economic theory:

$$x_i = x_i(p_i, m_i, z_i)$$

- x_i is $G \times 1$ vector of quantities demanded
- p_i is $G \times 1$ vector of prices
- m_i is income
- z_i is a vector of individual characteristics related to preferences

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Suppose a sample of one observation of n individuals' demands at time period t (this is a *cross section*). The model is not estimable as it stands.

- The form of the demand function is different for all i.
- Some components of z_i are subject to fluctuations that are not observable to outside modeler (people don't eat the same lunch every day). Break z_i into the observable components w_i and an unobservable component ε_i.

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An estimable (*e.g.*, econometric) model is

$$x_i = \beta_0 + p_i'\beta_p + m_i\beta_m + w_i'\beta_w + \varepsilon_i$$

We have imposed a number of restrictions on the theoretical model:

- The functions $x_i(\cdot)$ which may differ for all i have been restricted to all belong to the same parametric family.
- Of all parametric families of functions, we have restricted the model to the class of linear in the variables functions.

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These are **very strong restrictions**, compared to the theoretical model. Furthermore, **these restrictions have no theoretical basis.** The validity of any results we obtain using this model will be contingent on these restrictions being correct. For this reason, *specification testing* will be needed, to check that the model seems to be reasonable. Only when we are convinced that the model is at least approximately correct should we use it for economic analysis. In the next sections we will obtain results supposing that the econometric model is correctly specified. Later we will examine the consequences of misspecification and see some methods for determining if a model is correctly specified.

Ordinary Least Squares

The classical linear model is based upon several assumptions.

1. **Linearity**: the model is a linear function of the parameter vector β_0 :

$$y_t = x_t'\beta_0 + \varepsilon_t,$$

or in matrix form,

$$y = \mathbf{X}\beta_0 + \varepsilon,$$

where y is $n \times 1$, $\mathbf{X} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}'$, where x_t is $K \times 1$, and β_0 and ε are conformable. The subscript "0" in β_0 means this is the true value of the unknown parameter. It will be suppressed when it's not necessary for clarity. Linear models are more general than they might first appear, since one can employ nonlinear transformations of the variables:

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$$\varphi_0(z_t) = \left[\begin{array}{cccc} \varphi_1(w_t) & \varphi_2(w_t) & \cdots & \varphi_p(w_t) \end{array} \right] \beta_0 + \varepsilon_t$$

(The $\phi_i()$ are known functions). Defining $y_t = \varphi_0(z_t)$, $x_{t1} = \varphi_1(w_t)$, etc. leads to a model in the form of equation (??). For example, the Cobb-Douglas model

$$z = A w_2^{\beta_2} w_3^{\beta_3} \exp(\varepsilon)$$

can be transformed logarithmically to obtain

$$\ln z = \ln A + \beta_2 \ln w_2 + \beta_3 \ln w_3 + \varepsilon.$$

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2. **IID mean zero errors:**

$$\mathcal{E}(\varepsilon) = 0$$

$$Var(\varepsilon) = \mathcal{E}(\varepsilon\varepsilon') = \sigma_0^2 I_n$$

3. **Nonstochastic, linearly independent regressors**

   (a) X has rank K

   (b) X is nonstochastic

   (c) $\lim_{n\to\infty} \frac{1}{n} X'X = Q_X$, a finite positive definite matrix.

4. **Normality (Optional)**: ε is normally distributed

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- Estimation by least squares

The objective is to gain information about the unknown parameters β_0 and σ_0^2.

$$\begin{aligned} \hat{\beta} &= \arg\min s(\beta) = \sum_{t=1}^{n} (y_t - x_t'\beta)^2 \\ s(\beta) &= (y - X\beta)'(y - X\beta) \\ &= y'y - 2y'X\beta + \beta'X'X\beta \\ &= \| y - X\beta \|^2 \end{aligned}$$

Ordinary Least Squares

This last expression makes it clear how the OLS estimator chooses $\hat{\beta}$: it minimizes the Euclidean distance between y and $X\beta$.

- To minimize the criterion $s(\beta)$, take the f.o.n.c. and set them to zero:

$$D_{\beta}s(\hat{\beta}) = -2X'y + 2X'X\hat{\beta} = 0$$

so

$$\hat{\beta} = (X'X)^{-1}X'y.$$

Ordinary Least Squares

- To verify that this is a minimum, check the s.o.s.c.:

$$D^2_{\beta} s(\hat{\beta}) = 2X'X$$

Since $\rho(X) = K$, this matrix is positive definite, since it's a quadratic form in a p.d. matrix (identity matrix of order n), so $\hat{\beta}$ is in fact a minimizer.

- The *fitted values* are in the vector $\hat{y} = X\hat{\beta}$.
- The *residuals* are in the vector $\hat{\varepsilon} = y - X\hat{\beta}$
  - Note that

$$\begin{aligned} y &= X\beta + \varepsilon \\ &= X\hat{\beta} + \hat{\varepsilon} \end{aligned}$$

Ordinary Least Squares

- Estimating the error variance

The OLS estimator of σ_0^2 is

$$\widehat{\sigma_0^2} = \frac{1}{n-K}\hat{\varepsilon}'\hat{\varepsilon}$$

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- Goodness of fit

The fitted model is

$$y = X\hat{\beta} + \hat{\varepsilon}$$

Take the inner product:

$$y'y = \hat{\beta}'X'X\hat{\beta} + 2\hat{\beta}'X'\hat{\varepsilon} + \hat{\varepsilon}'\hat{\varepsilon}$$

But the middle term of the RHS is zero since $X'\hat{\varepsilon} = 0$, so

$$y'y = \hat{\beta}'X'X\hat{\beta} + \hat{\varepsilon}'\hat{\varepsilon}$$

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The *uncentered* R_u^2 is defined as

$$\begin{aligned} R_u^2 &= 1 - \frac{\hat{\varepsilon}'\hat{\varepsilon}}{y'y} \\ &= \frac{\hat{\beta}'X'X\hat{\beta}}{y'y} \\ &= \frac{\| P_X y \|^2}{\| y \|^2} \\ &= \cos^2(\phi), \end{aligned}$$

where ϕ is the angle between y and the span of X *(show with the one regressor, two observation example).*

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- The uncentered R^2 changes if we add a constant to y, since this changes ϕ. Another, more common definition measures the contribution of the variables, other than the constant term, to explaining the variation in y.
  - Let $\iota = (1, 1, ..., 1)'$, a n -vector. So

$$\begin{aligned} M_\iota &= I_n - \iota(\iota'\iota)^{-1}\iota' \\ &= I_n - \iota\iota'/n \end{aligned}$$

$M_\iota y$ just returns the vector of deviations from the mean.

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The *centered* R_c^2 is defined as

$$R_c^2 = 1 - \frac{\hat{\varepsilon}'\hat{\varepsilon}}{y'M_\iota y} = 1 - \frac{ESS}{TSS}$$

Supposing that X contains a column of ones (*i.e.*, there is a constant term),

$$X'\hat{\varepsilon} = 0 \Rightarrow \sum_t \hat{\varepsilon}_t = 0$$

Ordinary Least Squares

so $M_\iota\hat{\varepsilon} = \hat{\varepsilon}$. In this case

$$y'M_\iota y = \hat{\beta}'X'M_\iota X\hat{\beta} + \hat{\varepsilon}'\hat{\varepsilon}$$

So

$$R_c^2 = \frac{RSS}{TSS}$$

- Supposing that a column of ones is in the space spanned by X ($P_X\iota = \iota$), then one can show that $0 \leq R_c^2 \leq 1$.

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- Normality

$$\hat{\beta} = \beta_0 + (X'X)^{-1}X'\varepsilon$$

This is a linear function of ε, which is normally distributed. Therefore

$$\hat{\beta} \sim N\left(\beta_0, (X'X)^{-1}\sigma_0^2\right)$$

Ordinary Least Squares

- Efficiency (Gauss-Markov theorem)

The OLS estimator is a *linear estimator*, which means that it is a linear function of the dependent variable, y.

$$\hat{\beta} = [(X'X)^{-1}X']y$$
$$= Cy$$

It is also *unbiased*, as we proved above. One could consider other weights W in place of the OLS weights. We'll still insist upon unbiasedness. Consider $\tilde{\beta} = Wy$. If the estimator is unbiased

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$$
\begin{aligned}
\mathcal{E}(Wy) &= \mathcal{E}(WX\beta_0 + W\varepsilon) \\
&= WX\beta_0 \\
&= \beta_0 \\
&\Rightarrow \\
WX &= I_K
\end{aligned}
$$

Ordinary Least Squares

The variance of $\tilde{\beta}$ is

$$V(\tilde{\beta}) = WW'\sigma_0^2.$$

Define

$$D = W - (X'X)^{-1}X'$$

so

$$W = D + (X'X)^{-1}X'$$

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Since $WX = I_K$, $DX = 0$, so

$$\begin{aligned} V(\tilde{\beta}) &= \left(D + (X'X)^{-1}X'\right)\left(D + (X'X)^{-1}X'\right)' \sigma_0^2 \\ &= \left(DD' + (X'X)^{-1}\right)\sigma_0^2 \end{aligned}$$

So

$$V(\tilde{\beta}) \geq V(\hat{\beta}).$$

This is a proof of the Gauss-Markov Theorem.

Ordinary Least Squares

Theorem 1 (Gauss-Markov) *Under the classical assumptions, the variance of any linear unbiased estimator minus the variance of the OLS estimator is a positive semidefinite matrix.*

- It is worth noting that we have not used the normality assumption in any way to prove the Gauss-Markov theorem, so it is valid if the errors are not normally distributed, as long as the other assumptions hold.

Before considering the asymptotic properties of the OLS estimator it is useful to review the MLE estimator, since under the assumption of normal errors the two estimators coincide.

QUESTIONS!


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